Bond chaos in the Sherrington–Kirkpatrick model

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Abstract
We calculate the probability distribution of the overlap between a spin glass
and a copy of itself in which the bonds are randomly perturbed in varying
degrees. The overlap distribution is shown to go to a δ distribution in the
thermodynamic limit for arbitrarily small perturbations (bond chaos) and we
obtain the scaling behaviour of the distribution with system size N in the high-
and low-temperature phases and exactly at the critical temperature. The results
are relevant for the free energy fluctuations in the Sherrington–Kirkpatrick
model [1].

1. Introduction

Chaos is one of the fascinating properties of spin glasses. Whenever a spin glass is subjected
to a change (e.g. a change in temperature or in the coupling constants), the equilibrium state
of the system changes completely (in the thermodynamic limit), no matter how small the
change is. This behaviour has been first observed in hierarchical models [2] and was then
suggested for finite-dimensional spin glasses [3] within the droplet theory. Chaos is however
not restricted to replica symmetric droplet-like scenarios: in [4] it was shown that temperature
chaos exists in the Sherrington–Kirkpatrick model [5] which breaks the replica symmetry. In
this paper we show that bond chaos also exists in the Sherrington–Kirkpatrick model. This
is not surprising since bond chaos is usually a stronger effect than temperature chaos [6].
However, it has recently been found that an intricate connection exists between bond chaos
and the sample-to-sample fluctuations of the free energy [1, 7], which is a long-standing
problem in spin glass and extreme value theory. It is therefore necessary to calculate bond
chaos in order to make progress on fluctuations, and in this paper we will present the details
of the calculation which was only sketched in [1].

Temperature chaos has been simulated extensively in the literature, see e.g. [8] and
references therein. Bond chaos has not been simulated quite so much but nevertheless
for many different models, including the Edwards–Anderson model in dimensions 1–4
[6, 8–12], on the hierarchical Berker lattice [13, 14], for the Viana–Bray model [9] and
on Bethe lattices [11]. However, to the best of our knowledge, it has been simulated only

once for the Sherrington–Kirkpatrick model [9], using a form of bond chaos which is slightly different from that we consider here. In this paper we will show that the probability distribution of the overlap, which is our main quantity of interest, has a relatively complicated finite size scaling behaviour (see equation (68) below).

    This paper is organized as follows. In section 2 we specify the model and method we will be using and explain in section 3 how to obtain the probability distribution of the overlap for this model. We then proceed with the replica calculation in section 4 where we explicitly calculate this distribution above, at and below the critical temperature. We end with a conclusion in section 5.

2. Model

We will compare two real replicas of a Sherrington–Kirkpatrick spin glass with Hamiltonian

$$\mathcal{H}_\epsilon = -\frac{1}{\sqrt{N}} \sum_{i<j} K_{ij}(\epsilon) s_i s_j, \tag{1}$$

where s_i $(i = 1, \ldots, N)$ are N Ising spins and

$$K_{ij}(\epsilon) = \frac{1}{\sqrt{1+\epsilon^2}} J_{ij} + \frac{\epsilon}{\sqrt{1+\epsilon^2}} J'_{ij} \tag{2}$$

are Gaussian random variables of unit variance, composed of independent Gaussian random variables J_{ij} and J'_{ij}, also of unit variance. The parameter ϵ is a measure of 'distance' between the sets of bonds $\{K_{ij}(0)\}$ and $\{K_{ij}(\epsilon)\}$. If $\epsilon = 0$, the bonds are equal; if $\epsilon = \infty$, the bonds are completely uncorrelated.

    Our main question concerns the disorder averaged probability distribution of the spin–spin overlap $P_\epsilon(q)$ between the two replicas, the first with $\mathcal{H}_0$ and the other with $\mathcal{H}_\epsilon$. Here q is defined by

$$q = \frac{1}{N} \sum_i s_i^{1,0} s_i^{2,\epsilon}, \tag{3}$$

where $s_i^{r,x}$ is the ith spin in replica r, which has Hamiltonian $\mathcal{H}_x$.

    We know that for temperature chaos (i.e. when comparing two replicas with identical bonds but at different temperatures), even an infinitesimal temperature difference ΔT leads to $P_{\Delta T}(q) = \delta(q)$ *in the thermodynamic limit* [4]. This means that equilibrium states in the two replicas are totally uncorrelated. Since bond chaos is usually a stronger effect than temperature chaos, we expect the same here, and we will show this explicitly below. More interesting, however, and also more important, is the question of how $P_\epsilon(q)$ scales towards the δ function with system size N. This information is for instance needed for the calculation of the sample-to-sample free energy fluctuations [1, 7].

3. Probability distribution of the overlap

In order to calculate $P_\epsilon(q)$ we first try to formally calculate $P_{\epsilon,J}(q)$, the nonaveraged probability distribution to find the overlap q. Here and in the following, the subscript J indicates a nonaveraged quantity. Given a realization of the disorder for the two replicas, we can write a partition function $Z_{\epsilon,J}(q)$ for them, constrained to have the overlap q,

$$Z_{\epsilon,J}(q) = \text{Tr}\, \delta\left(q - \frac{1}{N} \sum_i s_i^{1,0} s_i^{2,\epsilon}\right) \exp\left(\beta \sum_{i<j} K_{ij}(0) s_i^{1,0} s_j^{1,0} + \beta \sum_{i<j} K_{ij}(\epsilon) s_i^{2,\epsilon} s_j^{2,\epsilon}\right). \tag{4}$$

This method of constraining two systems to have a given overlap was first suggested in [15]. From this, one gets the free energy $\beta F_{\epsilon,J}(q) = -\log Z_{\epsilon,J}(q)$, and defining $Y_{\epsilon,J} = \int_0^1 dq\, Z_{\epsilon,J}(q)$, the probability distribution of q is given by a Boltzmann factor,

$$P_{\epsilon,J}(q) = \frac{e^{-\beta F_{\epsilon,J}(q)}}{Y_{\epsilon,J}} = \frac{Z_{\epsilon,J}(q)}{Y_{\epsilon,J}}. \tag{5}$$

Of course, we cannot calculate $F_{\epsilon,J}(q)$ for a given disorder but we can calculate its disorder average $F_{\epsilon}(q) = E F_{\epsilon,J}$ (the disorder average is denoted by the symbol $E\ldots$) in the thermodynamic limit by replica methods. This will be done below. But first we continue formally in order to assess the approximations we are going to make.

We split $F_{\epsilon,J}(q)$ into three parts,

$$F_{\epsilon,J}(q) = N f_{\epsilon}(q) + \Delta F_{\epsilon}(q) + \Delta F_{\epsilon,J}(q). \tag{6}$$

The first part, $N f_{\epsilon}(q)$, is the extensive part of the average free energy. The second part, $\Delta F_{\epsilon}(q)$, is the finite size correction to it. Finally, $\Delta F_{\epsilon,J}(q)$ is the fluctuation of the free energy about its disorder average $F_{\epsilon}(q)$, so $E \Delta F_{\epsilon,J}(q) = 0$.

In order to calculate $P_{\epsilon}(q) = E P_{\epsilon,J}(q)$, we first look at $Y_{\epsilon,J}$. It can be written as follows,

$$
\begin{aligned}
Y_{\epsilon,J} &= \int_0^1 dq\, e^{-\beta N f_{\epsilon}(q) - \beta(\Delta F_{\epsilon}(q) + \Delta F_{\epsilon,J}(q))} \\
&= \int_0^1 dq\, e^{-\beta N f_{\epsilon}(q)} \left(1 + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} \frac{\int_0^1 dq\, e^{-\beta N f_{\epsilon}(q)} (\Delta F_{\epsilon}(q) + \Delta F_{\epsilon,J}(q))^n}{\int_0^1 dq\, e^{-\beta N f_{\epsilon}(q)}} \right) \\
&= Y_{\epsilon}^0 \left(1 + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} [(\Delta F_{\epsilon} + \Delta F_{\epsilon,J})^n]_0 \right),
\end{aligned}
\tag{7}
$$

where $Y_{\epsilon}^0 := \int_0^1 dq\, e^{-\beta N f_{\epsilon}(q)}$ and $[\cdots]_0$ is the average taken with the probability distribution

$$P_{\epsilon}^0(q) = \frac{e^{-\beta N f_{\epsilon}(q)}}{Y_{\epsilon}^0}. \tag{8}$$

We then have

$$\frac{1}{Y_{\epsilon,J}} = \frac{1}{Y_{\epsilon}^0} \int_0^{\infty} dx\, e^{-x(1 + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} [(\Delta F_{\epsilon} + \Delta F_{\epsilon,J})^n]_0)} \tag{9}$$

and

$$P_{\epsilon}(q) = P_{\epsilon}^0(q) E \int_0^{\infty} dx\, e^{-x - x \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} [(\Delta F_{\epsilon} + \Delta F_{\epsilon,J})^n]_0 - \beta(\Delta F_{\epsilon}(q) + \Delta F_{\epsilon,J}(q))} \tag{10}$$

$$
\begin{aligned}
= P_{\epsilon}^0(q) \Bigg(&1 - \beta(\Delta F_{\epsilon}(q) - [\Delta F_{\epsilon}]_0) \\
&- \frac{\beta^2}{2} [(\Delta F_{\epsilon}(q) - [\Delta F_{\epsilon}]_0)^2]_0 - \frac{\beta^2}{2} E[(\Delta F_{\epsilon,J}(q) - [\Delta F_{\epsilon,J}]_0)^2]_0 \\
&+ \frac{\beta^2}{2} (\Delta F_{\epsilon}(q) - [\Delta F_{\epsilon}]_0)^2 + \frac{\beta^2}{2} E(\Delta F_{\epsilon,J}(q) - [\Delta F_{\epsilon,J}]_0)^2 + \cdots \Bigg).
\end{aligned}
\tag{11}
$$

The last line follows from expanding the exponentials and sorting by powers of ΔF_{ϵ} and $\Delta F_{\epsilon,J}$.

Equation (11) shows precisely what can be expected of the calculation which is to follow. We will calculate the first term of this expression, $P_{\epsilon}^0(q)$. This is the same term one would

have written from the start by following large deviation statistics principles. Anything beyond this would require us to calculate the finite size corrections $\Delta F_\epsilon(q)$ and $\Delta F_{\epsilon,J}(q)$ which is impossible due to the massless modes present in the spin glass phase. But our precise analysis allows us to gauge the applicability of the large deviation statistics approximation: although $\Delta F_\epsilon(q)$ and $\Delta F_{\epsilon,J}(q)$ are not necessarily small (they grow with some subdominant power of N), they only appear in the combinations $\Delta F_\epsilon(q) - [\Delta F_\epsilon]_0$ and $\Delta F_{\epsilon,J}(q) - [\Delta F_{\epsilon,J}]_0$. If these differences are small, our results not only hold for 'large' deviations but also for 'small' ones. While we have no proof for this, application of our results to the free energy fluctuations above and at the critical temperature [1, 7] shows that it is at least true at those temperatures. As to the low-temperature phase, we will see later that for $\epsilon = 0$ and q less than the Edwards–Anderson order parameter q_{EA}, the corrections are of order 1 but small enough not to introduce any qualitative changes, and we expect the same to hold for $\epsilon > 0$.

4. Replica calculation

Temperature chaos in mean-field spin glasses has been treated in the literature [4, 16] and we refer the reader to these papers for details. Repeating Rizzo's calculation [16] (see also [17] for the calculation at $\epsilon = 0$) but for bond chaos rather than temperature chaos, one arrives at the following expression for the disorder averaged, replicated, constrained partition function $E Z_{\epsilon,J}^n(q)$

$$
E Z_{\epsilon,J}^n(q) = \int \left(\prod_{\alpha=1}^n dz_\alpha \right) \left(\prod_{\alpha<\beta}^{2n} dT_{\alpha\beta} \right) \exp \left(2N \left[\frac{\tau}{2} \sum_{\alpha\beta}^n Q_{\alpha\beta}^2 + \frac{\tau'}{2} \sum_{\alpha\beta}^n R_{\alpha\beta}^2 \right.\right.
$$
$$
+ \frac{w}{6} \left(\sum_{\alpha\beta\gamma}^n Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha} + 3 \sum_{\alpha\beta\gamma}^n Q_{\alpha\beta} R_{\beta\gamma} R_{\gamma\alpha} \right) + \frac{y}{12} \left(\sum_{\alpha\beta}^n Q_{\alpha\beta}^4 + \sum_{\alpha\beta}^n R_{\alpha\beta}^4 \right)
$$
$$
\left.\left. - q \sum_{\alpha=1}^n z_\alpha + \frac{\sqrt{1+\epsilon^2}}{4} \sum_{\alpha=1}^n \left(R_{\alpha\alpha}^2 - (R_{\alpha\alpha} - 2z_\alpha)^2 \right) \right] \right). \tag{12}
$$

Here, $\tau = \frac{1}{2}(1 - 1/\beta^2)$, $\tau' = \frac{1}{2}\left(1 - \sqrt{1+\epsilon^2}/\beta^2\right)$ and the $2n \times 2n$ matrix T is given by

$$
T = \begin{pmatrix} Q & R \\ R & Q \end{pmatrix} \tag{13}
$$

and the $n \times n$ matrix Q is zero on the diagonal, while R, also of size $n \times n$, is not. The latter will therefore be split into a part on the diagonal $p_d \mathbf{1}$ and the rest on P. At this stage it is an ansatz to make all entries on the diagonal of R equal to p_d but this need not be done and one could proceed without this assumption and justify it later.

Note that, in contrast to [16], we are using the truncated model here by keeping only those fourth-order terms which are written in equation (12). This will make the calculations below much easier.

The saddle point equations following from this expression can easily be derived and, making a Parisi symmetry breaking ansatz for P and Q, one gets the following system of equations:

$$
0 = \tau q(x) + \frac{y}{3} q^3(x) - \frac{w}{2} \int_0^x dz (q(x) - q(z))^2 - \frac{w}{2} \int_0^x dz (p(x) - p(z))^2
$$
$$
- w(\overline{p} - p_d) p(x) - w \overline{q} q(x) \tag{14}
$$

$$0 = \tau' p(x) + \frac{y}{3} p^3(x) - w \int_0^x \mathrm{d}z (p(x) - p(z))(q(x) - q(z))$$
$$- w(\overline{p} - p_d)q(x) - w\overline{q}p(x) \tag{15}$$

$$q = p_d + \frac{2y}{3} p_d^3 - 2w \int_0^1 \mathrm{d}z \, p(z)q(z). \tag{16}$$

The variable q (without argument) in the last equation is the overlap the two real replicas are forced to have. As usual, $\overline{p}$ and $\overline{q}$ denote the integrals over $p(x)$ and $q(x)$ from 0 to 1. The free energy per spin may be expressed in terms of the solutions $p(x)$, $q(x)$ and p_d of these equations and is given by

$$\beta f_\epsilon(q) = q p_d - \frac{p_d^2}{2} - \frac{y p_d^4}{6} - \frac{q^2}{2(1 - 2\tau')} + \tau \int_0^1 \mathrm{d}z \, q^2(z) + \frac{y}{6} \int_0^1 \mathrm{d}z \, q^4(z)$$
$$- \frac{w}{3} \int_0^1 \mathrm{d}z \, z q^3(z) - w \int_0^1 \mathrm{d}z \, q^2(z) \int_z^1 \mathrm{d}z' q(z') + 2w p_d \int_0^1 \mathrm{d}z \, p(z)q(z)$$
$$+ \tau' \int_0^1 \mathrm{d}z \, p^2(z) + \frac{y}{6} \int_0^1 \mathrm{d}z \, p^4(z) - w \int_0^1 \mathrm{d}z \, p^2(z) \int_z^1 \mathrm{d}z' q(z')$$
$$- 2w \int_0^1 \mathrm{d}z \, p(z)q(z) \int_z^1 \mathrm{d}z' p(z') - w \int_0^1 \mathrm{d}z \, z p^2(z)q(z). \tag{17}$$

It can be checked that this free energy gives back the saddle point equations (14)–(16) when the derivatives with respect to $p(x)$, $q(x)$ and p_d are taken.

4.1. Above the critical temperature

Above the critical temperature, $\tau < 0$, the saddle point equations can be solved perturbatively in the limit of small q. For $q = 0$, the exact solution is $q(x) = p(x) = 0$ and $p_d = 0$. For $0 < q \ll |\tau|$ it is easy to see that $q(x) = \mathcal{O}(q^2)$, $p(x) = \mathcal{O}(q^2)$ and $p_d = q + \mathcal{O}(q^3)$. Plugging this into the free energy, equation (17), yields

$$\beta f_\epsilon(q) = \frac{q^2}{2} \left(1 - \frac{\beta^2}{\sqrt{1 + \epsilon^2}} \right) + \mathcal{O}(q^4). \tag{18}$$

The probability distribution $P_\epsilon^0(q)$ is thus

$$P_\epsilon^0(q) \propto \mathrm{e}^{-N(\frac{q^2}{2} h(\epsilon) + \mathcal{O}(q^4))} \tag{19}$$

with $h(\epsilon) = 1 - \frac{\beta^2}{\sqrt{1+\epsilon^2}}$.

4.2. At the critical temperature

At the critical temperature, $\tau = 0$, we can first solve the saddle point equations in the limit $q \ll |\tau'|$. Just like above the critical temperature, the solution is $q(x) = \mathcal{O}(q^2)$, $p(x) = \mathcal{O}(q^2)$ and $p_d = q + \mathcal{O}(q^3)$, and the free energy is also given by equation (18) in this limit. Expanding in powers of ϵ, we find

$$\beta f_\epsilon(q) = \frac{q^2}{2} h(\epsilon) = \frac{q^2 \epsilon^2}{4} + \mathcal{O}(\epsilon^4) \qquad (q \ll \epsilon^2 \ll 1). \tag{20}$$

We will also need the solution of the saddle point equations in a different limit, namely $\epsilon^2 \ll q \ll 1$. To obtain this, it is convenient to rewrite the saddle point equations and the

free energy in terms of the functions $a(x) := q(x) + p(x)$ and $b(x) := q(x) - p(x)$. A straightforward calculation leads to

$$0 = (\tau + wp_d)a(x) - \frac{\tau - \tau'}{2}(a(x) - b(x)) + \frac{y}{12}(a^3(x) + 3a(x)b^2(x))$$
$$- \frac{w}{2}\int_0^x dz(a(x) - a(z))^2 - w\bar{a}a(x) \tag{21}$$

$$0 = (\tau - wp_d)b(x) + \frac{\tau - \tau'}{2}(a(x) - b(x)) + \frac{y}{12}(b^3(x) + 3b(x)a^2(x))$$
$$- \frac{w}{2}\int_0^x dz(b(x) - b(z))^2 - w\bar{b}b(x) \tag{22}$$

$$q = p_d + \frac{2y}{3}p_d^3 - \frac{w}{2}\int_0^1 dz(a^2(z) - b^2(z)) \tag{23}$$

and

$$\beta f_\epsilon(q) = qp_d - \frac{p_d^2}{2} - \frac{yp_d^4}{6} - \frac{q^2}{2(1 - 2\tau')} + \frac{\tau}{2}\int_0^1 dz(a^2(z) + b^2(z))$$
$$- \frac{\tau - \tau'}{4}\int_0^1 dz(a(z) - b(z))^2 - \frac{w}{6}\int_0^1 dz\, z(a^3(z) + b^3(z))$$
$$- \frac{w}{2}\int_0^1 dz\int_z^1 dz'(a^2(z)a(z') + b^2(z)b(z'))$$
$$+ \frac{y}{48}\int_0^1 dz(a^4(z) + 6a^2(z)b^2(z) + b^4(z)) + \frac{wp_d}{2}\int_0^1 dz(a^2(z) - b^2(z)). \tag{24}$$

These saddle point equations can easily be solved for $\tau = \tau' = 0$, and the solution is

$$a(x) = \begin{cases} \dfrac{2wx}{y} & x < x_2' \\ \dfrac{2wx_2'}{y} & x \geq x_2' \end{cases} \tag{25}$$

$$b(x) = 0 \tag{26}$$

$$q = p_d + \frac{2y}{3}p_d^3 - \frac{2w^3}{y^2}\left(x_2'^2 - \frac{2}{3}x_2'^3\right). \tag{27}$$

The breakpoint of $a(x)$ is $x_2' = 1 - \sqrt{1 - \frac{yp_d}{w}}$. The value of p_d has to be obtained by solving equation (27) for p_d.

When we want to find the free energy perturbatively in the limit $0 < |\tau'| \ll q \ll 1$, we note that $\Delta a(x)$ and $\Delta b(x)$ (the correction terms for nonzero τ') are both of order τ'. Since these are the corrections to the saddle point solution found at $\tau' = 0$, they only contribute to the free energy to order τ'^2 and can therefore be neglected as long as we are only interested in the free energy to order τ'. Plugging the functions $a(x)$ and $b(x)$ just found into equation (24) yields

$$\beta f_\epsilon(q) = \frac{w}{6}q^3 - \frac{3}{4}\tau'q^2 + \mathcal{O}(\tau'^2) + \mathcal{O}(q^4)$$
$$= \frac{w}{6}q^3 + \frac{3\epsilon^2}{16}q^2 + \mathcal{O}(\epsilon^4, q^4) \qquad (\epsilon^2 \ll q \ll 1). \tag{28}$$

We note that for $\epsilon \ll N^{-1/6}$ the exponent $N\beta f_\epsilon(q)$ of the probability distribution $P_\epsilon^0(q) \sim e^{-N\beta f_\epsilon(q)}$ is dominated by $Nwq^3/6$ from equation (28), while for $N^{-1/6} \ll \epsilon \ll 1$ it is dominated by $N\frac{q^2}{2}h(\epsilon)$ from equation (20). Therefore the probability distribution is

$$
P_\epsilon^0(q) \propto \begin{cases} e^{-Nwq^3/6} & \epsilon \ll N^{-1/6} \\ e^{-Nq^2h(\epsilon)/2} & N^{-1/6} \ll \epsilon. \end{cases} \tag{29}
$$

4.3. Below the critical temperature

4.3.1. Some exact solutions of the saddle point equations. Equations (14)–(16) cannot be solved for general q and ϵ (or rather τ' which contains the only reference to ϵ). They can however be solved exactly for $\epsilon = 0$ and $q = 0$ and perturbatively in various limits.

The solution for $q = 0$ is simply

$$
p_1(x) = 0 \tag{30}
$$

$$
q_1(x) = \begin{cases} \dfrac{wx}{2y} & x < x_2 \\ \dfrac{wx_2}{2y} & x \geqslant x_2 \end{cases} \tag{31}
$$

$$
p_d = 0 \tag{32}
$$

where $x_2 = 1 - \sqrt{1 - \frac{4y\tau}{w^2}}$. That this is a solution can easily be checked since equation (16) is clearly satisfied, equation (15) is also satisfied by $p_d = 0$ and $p(x) = 0$, irrespective of $q(x)$, and equation (14) in this case reduces to the normal Parisi equation the solution of which is well known and given in equation (31).

The solution for $\tau = \tau'$ and $q < q_{EA}$, where q_{EA} is the Edwards–Anderson order parameter, has been found by Rizzo [16] and is given by

$$
p_2(x) = \begin{cases} \dfrac{wx}{y} & x < x_1 \\ \dfrac{wx_1}{y} & x \geqslant x_1 \end{cases} \tag{33}
$$

$$
q_2(x) = \begin{cases} \dfrac{wx}{y} & x < x_1 \\ \dfrac{wx_1}{y} & x_1 \leqslant x < 2x_1 \\ \dfrac{wx}{2y} & 2x_1 \leqslant x < x_2 \\ \dfrac{wx_2}{2y} & x_2 \leqslant x \leqslant 1 \end{cases} \tag{34}
$$

$$
p_d = \frac{q}{1 - 2\tau} \tag{35}
$$

with $x_1 = \frac{yp_d}{w}$. This solution however only exists if $2x_1 < x_2$, which is the case when $q < q_{EA}$. In the limit $q \to 0$ it goes to the solution found above for $q = 0$. As shown by Rizzo, the free energy βf_0 of this solution (for any q where this solution exists) is precisely equal to that of two unconstrained systems; we may therefore use βf_0 as reference energy.

For $\tau = \tau'$ and $q > q_{EA}$, another solution takes over [17]. It is given by

$$p_3(x) = q_3(x) = \begin{cases} \dfrac{wx}{y} & x < x_3 \\[2mm] \dfrac{wx_3}{y} & x \geqslant x_3 \end{cases} \tag{36}$$

$$q = p_d + \frac{2y}{3} p_d^3 - \frac{2w^3}{y^2}\left(x_3^2 - \frac{2}{3}x_3^3\right). \tag{37}$$

The breakpoint is $x_3 = 1 - \sqrt{1 - \frac{y\tau + wyp_d}{w^2}}$, and p_d can be obtained by solving equation (37) for p_d. Since the details can be found in [17], we merely quote here that the excess free energy of this solution is

$$\beta f - \beta f_0 = c_0(q - q_{EA})^3 \tag{38}$$

with some positive constant c_0.

4.3.2. Perturbative solution in the limit $\epsilon \to \infty$.   The first limiting case to consider is $\epsilon \to \infty$ or, equivalently, $\tau' \to -\infty$. In this case, dividing equation (15) by τ' yields to leading order

$$p(x) = \frac{wp_d q(x)}{-\tau'} + \mathcal{O}(1/\tau'^2). \tag{39}$$

Inserting this into equation (14) gives, again to leading order,

$$0 = \left(\tau + \frac{w^2 p_d^2}{-\tau'}\right)q(x) - \frac{y}{3}q^3(x) - \frac{w}{2}\int_0^1 \mathrm{d}z(q(x) - q(z))^2 - w\bar{q}q(x), \tag{40}$$

i.e. precisely the Parisi equation but for a slightly *lower* temperature $\tau + \frac{w^2 p_d^2}{-\tau'}$.

Let us look at the free energy of this solution in the limit $\tau' = -\infty$. From equation (16) we get $qp_d = p_d^2 + 2yp_d^4/3$ and the free energy difference $\beta f_\infty(q) - \beta f_0$ is

$$\beta f_\infty(q) - \beta f_0 = \frac{p_d^2}{2} + \frac{yp_d^4}{2} = \frac{q^2}{2} + \mathcal{O}(q^4). \tag{41}$$

This means that the probability density $P_\infty^0(q)$ is proportional to $\exp(-N(q^2/2 + \mathcal{O}(q^4)))$, from which $[q^2]_0$ can be evaluated and is given by $[q^2]_0 = \frac{1}{N}$. This is a useful result to check whether the method works since it is easy to convince oneself that this is indeed true for two replicas with completely uncorrelated couplings:

$$E\langle q^2\rangle = E\left\langle\left(\frac{1}{N}\sum_i s_i t_i\right)^2\right\rangle = E\left\langle\frac{2}{N^2}\left(\sum_{i<j} s_i s_j t_i t_j + \frac{N}{2}\right)\right\rangle = \frac{1}{N}, \tag{42}$$

in agreement with the replica result.

4.3.3. Perturbative solution for $\epsilon^2 \ll q \ll 1$.   We now turn to the opposite limit, $\epsilon \to 0$ or $\tau' \to \tau$. We first consider the case $\epsilon^2 \ll q \ll 1$. In order to solve the saddle point equations in this limit, it is again convenient to consider the version in terms of $a(x)$ and $b(x)$, equations (21)–(23). The solution for $\tau' = \tau$, i.e. $\Delta\tau = \tau - \tau' = 0$, and arbitrary q (small enough such that the solution exists), taken from equations (33)–(35), is

$$p_d = \frac{q}{1 - 2\tau} \tag{43}$$

$$a(x) = \begin{cases} \dfrac{2wx}{y} & x < x_1 \\[2mm] \dfrac{2wx_1}{y} & x_1 \leqslant x < 2x_1 \\[2mm] \dfrac{w(x + 2x_1)}{2y} & 2x_1 \leqslant x < x_2 \\[2mm] \dfrac{w(x_2 + 2x_1)}{2y} & x_2 \leqslant x \leqslant 1 \end{cases} \tag{44}$$

$$b(x) = \begin{cases} 0 & x < 2x_1 \\[2mm] \dfrac{w(x - 2x_1)}{2y} & 2x_1 \leqslant x < x_2 \\[2mm] \dfrac{w(x_2 - 2x_1)}{2y} & x_2 \leqslant x \leqslant 1. \end{cases} \tag{45}$$

Surprisingly, a perturbative expansion of the saddle point equations for small $\Delta\tau$ shows that the leading order correction to this solution is of order $\sqrt{\Delta\tau}$. It is straightforward to show that it is given by

$$\Delta a(x) = 0 \tag{46}$$

$$\Delta b(x) = \begin{cases} 0 & x < x_1 \\[2mm] \sqrt{\dfrac{2\Delta\tau}{3y}} & x_1 < x < 2x_1 \\[2mm] 0 & 2x_1 < x \leqslant 1. \end{cases} \tag{47}$$

Inserting this into the free energy difference leads to

$$\beta f_\epsilon(q) - \beta f_0 = \frac{y p_d^3}{3w}\Delta\tau = \frac{y\epsilon^2 q^3}{12w\beta^2(1 - 2\tau)^3} =: c_1\epsilon^2 q^3 \qquad (\epsilon^2 \ll q \ll 1), \tag{48}$$

defining the constant c_1.

4.3.4. Perturbative solution for $q \ll \epsilon^2 \ll 1$. In this limit we can use the solution for $q = 0$ from equations (30)–(32) as reference solution and construct a perturbative solution for small q (or p_d) from it, i.e. we set $p(x) = p_1(x) + \Delta p(x) = \Delta p(x)$ and $q(x) = q_1(x) + \Delta q(x)$ and write the lowest order in p_d from equation (15):

$$0 = \tau'\Delta p(x) - w\int_0^x dz(\Delta p(x) - \Delta p(z))(q_1(x) - q_1(z)) \\ - w(\overline{\Delta p} - p_d)q_1(x) - w\overline{q_1}\Delta p(x). \tag{49}$$

As we can see, $\Delta q(x)$ does not appear in this equation at all, and since $p(x)$ only appears quadratically or in combination with p_d in equation (14), we conclude that $\Delta q(x)$ must be in fact of higher order $\left(\mathcal{O}(p_d^2)\right)$ such that we may set it equal to zero.

Equation (49) can be solved in the following way. First note that $\overline{q_1} = \tau/w$ such that

$$0 = -\Delta\tau\Delta p(x) - w\int_0^x dz(\Delta p(x) - \Delta p(z))(q_1(x) - q_1(z)) - w(\overline{\Delta p} - p_d)q_1(x). \tag{50}$$

Differentiating once with respect to x yields

$$0 = -\Delta\tau\Delta p'(x) - wq_1'(x)\int_0^x dz(\Delta p(x) - \Delta p(z)) \\ - w\Delta p'(x)\int_0^x dz(q_1(x) - q_1(z)) - w(\overline{\Delta p} - p_d)q_1'(x). \tag{51}$$

We conclude that $\Delta p'(x) = 0$ whenever $q_1'(x) = 0$. When $q_1'(x) \neq 0$, differentiating equation (51) once more gives (noting that $q_1'(x) = 0$)

$$0 = -\Delta\tau\,\Delta p''(x) - 2wxq_1'(x)\Delta p'(x) - w\Delta p''(x) \int_0^x \mathrm{d}z(q_1(x) - q_1(z)) \tag{52}$$

$$= -\left(\Delta\tau + \frac{w^2x^2}{4y}\right)\Delta p''(x) - \frac{w^2x}{y}\Delta p'(x). \tag{53}$$

This differential equation can easily be solved and one gets

$$\Delta p'(x) = C'\left(\Delta\tau + \frac{w^2x^2}{4y}\right)^{-2} \tag{54}$$

with an as yet undetermined constant C'. Integrating once with respect to x yields

$$\Delta p(x) = \frac{C'\sqrt{y}}{w\Delta\tau^{3/2}}f\left(\frac{wx}{2\sqrt{y\Delta\tau}}\right) =: Cf\left(\frac{wx}{2\sqrt{y\Delta\tau}}\right), \tag{55}$$

where

$$f(z) = \frac{z}{1+z^2} + \arctan z. \tag{56}$$

This is true for $x < x_2$ while for $x > x_2$, $\Delta p(x)$ stays constant. We can now calculate $\overline{\Delta p}$,

$$\overline{\Delta p} = C\int_0^{x_2}\mathrm{d}x f\left(\frac{wx}{2\sqrt{y\Delta\tau}}\right) + C(1-x_2)f\left(\frac{wx_2}{2\sqrt{y\Delta\tau}}\right) \tag{57}$$

$$= C\arctan(wx_2/2\sqrt{y\Delta\tau}) + C(1-x_2)\frac{wx_2/2\sqrt{y\Delta\tau}}{1+w^2x_2^2/4y\Delta\tau}. \tag{58}$$

The constant C can be determined from equation (51) by taking the limit $x \to 0$. In this limit the equation reads

$$\Delta\tau\,\Delta p'(0) = w(p_d - \overline{\Delta p})q_1'(0) \tag{59}$$

such that

$$\frac{Cw\sqrt{\Delta\tau}}{\sqrt{y}} = \frac{w^2}{2y}p_d - \frac{w^2}{2y}\overline{\Delta p} \tag{60}$$

which yields upon expanding in powers of $\sqrt{\Delta\tau}$,

$$C = \frac{2}{\pi}p_d + \frac{4}{\pi^2}p_d\left(\frac{2}{3} - x_2\right)\left(\frac{2\sqrt{y\Delta\tau}}{wx_2}\right)^3 + \mathcal{O}(\Delta\tau^{5/2}). \tag{61}$$

With this value of C, $p(x) = \Delta p(x)$ together with $q(x) = q_1(x)$ is the correct solution of equations (14) and (15) for small $\Delta\tau$ up to order p_d.

What of the free energy of this solution? In order to calculate this, we eliminate q from equation (17) using equation (16) and subtract βf_0 to get

$$\beta f_\epsilon(q) - \beta f_0 = \frac{p_d^2}{2} + \frac{yp_d^4}{2} - \frac{1}{2(1-2\tau')}\left(p_d + \frac{2yp_d^3}{3} - 2w\int_0^1\mathrm{d}z\,p(z)q_1(z)\right)^2$$

$$- \Delta\tau\int_0^1\mathrm{d}z\,p^2(z) + \frac{y}{6}\int_0^1\mathrm{d}z\,p^4(z) - w\int_0^1\mathrm{d}z\,p^2(z)\int_0^z\mathrm{d}z'(q_1(z) - q_1(z'))$$

$$- 2w\int_0^1\mathrm{d}z\,p(z)q_1(z)\int_z^1\mathrm{d}z'\,p(z'). \tag{62}$$

We can calculate the integrals appearing in this equation and obtain

$$\int_0^1 \mathrm{d}z\, p(z)q_1(z) = \frac{\tau p_d}{w} - \frac{\Delta\tau p_d}{w} + \frac{4\sqrt{y}\Delta\tau^{3/2}p_d}{\pi w^2} + \mathcal{O}(\Delta\tau^{5/2}) \tag{63}$$

$$\int_0^1 \mathrm{d}z\, p^2(z) = p_d^2 - \frac{6\sqrt{y}}{\pi w}p_d^2\sqrt{\Delta\tau} + \mathcal{O}(\Delta\tau^{3/2}) \tag{64}$$

and

$$\int_0^1 \mathrm{d}z\, p(z)\left(p(z)\int_0^z \mathrm{d}z'(q_1(z) - q_1(z')) + 2q_1(z)\int_z^1 \mathrm{d}z'\, p(z')\right)$$

$$= \frac{\tau}{w}p_d^2 - 2\frac{\Delta\tau}{w}p_d^2 + \frac{10\sqrt{y}\Delta\tau^{3/2}}{\pi w^2}p_d^2 + \mathcal{O}(\Delta\tau^{5/2}). \tag{65}$$

Assembling these results, the terms of order $\Delta\tau$ cancel and the free energy difference is

$$\beta f_\epsilon(q) - \beta f_0 = \frac{4\sqrt{y}}{\pi w}p_d^2 \Delta\tau^{3/2} + \mathcal{O}(\Delta\tau^3) = c_2\epsilon^3 q^2 + \mathcal{O}(\epsilon^5) \qquad \text{with}$$

$$c_2 = \frac{4\sqrt{y}}{\pi w}\frac{1}{(1-2\tau)^2}\frac{1}{8\beta^3}. \tag{66}$$

4.3.5. Perturbative solution for $q \ll \min(1, \epsilon^2)$. In the preceding subsection we have calculated the free energy for $q \ll \epsilon \ll 1$. The condition $\epsilon \ll 1$ is an unnecessary restriction, however, and one can in principle carry out the same calculation as above without expanding for small ϵ, as long as $q \ll \min(1, \epsilon^2)$. The result is

$$\beta f_\epsilon(q) - \beta f_0 = f(\epsilon)q^2 + \mathcal{O}(q^3), \tag{67}$$

where the function $f(\epsilon)$ is monotonic and has the properties $f(\epsilon) = c_2\epsilon^3 + \mathcal{O}(\epsilon^5)$ for $\epsilon \to 0$ and $f(\epsilon) \to \frac{1}{2}$ for $\epsilon \to \infty$. The latter follows from the solution for $\tau' \to -\infty$ which we found previously. As we would not need any more detailed information about $f(\epsilon)$, we will not carry out this tedious calculation in detail here.

4.3.6. Probability distribution. As before, we want to calculate the probability distribution $P_\epsilon^0(q) \sim \mathrm{e}^{-N\beta(f_\epsilon(q)-f_0)}$. We observe that the admissible range of ϵ (the interval from 0 to ∞) divides into four parts. For $\epsilon \ll N^{-1/2}$, both equations (48) and (66) produce a negligible exponent $N\beta(f_\epsilon(q) - f_0)$ for all $q \in [0, q_{\mathrm{EA}}]$. The probability distribution $P_\epsilon^0(q)$ is thus a constant in that interval, with an exponentially suppressed tail for $q > q_{\mathrm{EA}}$ from equation (38). In the range $N^{-1/2} \ll \epsilon \ll N^{-1/5}$ the exponent $N\beta(f_\epsilon(q) - f_0)$ is negligible for $q \ll \epsilon^2$ when equation (66) prevails. It only becomes noticeable when $q \geqslant N^{-2/5} \gg \epsilon^2$ such that we can approximate it by equation (48). For $N^{-1/5} \ll \epsilon \leqslant \epsilon_0$ with some arbitrary small and fixed ϵ_0 independent of N, the exponent is dominated by equation (66). Finally, for $\epsilon_0 < \epsilon$ we can use equation (67).

Combined, we can write

$$P_\epsilon^0(q) \propto \begin{cases} \hat\theta(q - q_{\mathrm{EA}}) & \epsilon \ll N^{-1/2} \\ \mathrm{e}^{-Nc_1\epsilon^2 q^3} & N^{-1/2} \ll \epsilon \ll N^{-1/5} \\ \mathrm{e}^{-Nc_2\epsilon^3 q^2} & N^{-1/5} \ll \epsilon \leqslant \epsilon_0 \\ \mathrm{e}^{-Nf(\epsilon)q^2} & \epsilon_0 < \epsilon, \end{cases} \tag{68}$$

where the function $\hat\theta(x) = 1$ for $x < 0$ and $\hat\theta(x) = \mathrm{e}^{-Nc_0 x^3}$ for $x > 0$. In principle, the latter two regimes could be combined to $P_\epsilon^0(q) \propto \mathrm{e}^{-Nf(\epsilon)q^2}$ for $N^{-1/5} \ll \epsilon$ but splitting them makes the dependence on ϵ more explicit.

A note is in order about the probability distribution at $\epsilon = 0$. Clearly, the result in equation (68) does not coincide with the known distribution from the Parisi solution [18–20], in particular the δ peak at q_{EA} is missing and in the range $0 \leqslant q < q_{EA}$ the probability distribution is flat. The latter is not only an artefact of using the truncated model (for which the distribution would indeed be flat) but was already derived in [16, 17] for the full model. Both the flatness and the missing δ peak originate from neglecting the finite size corrections in equation (11). We conclude that these corrections are of order 1 for $q < q_{EA}$ (but do not change the probability distribution *qualitatively*, i.e. do not introduce gaps or zeros in the distribution) and conspire to form the δ peak for large N at $q = q_{EA}$. When $\epsilon > N^{-1/2}$ we do not expect any δ peaks since the equilibrium states in the two replicas start to differ substantially, and in the light of what we saw at $\epsilon = 0$ we expect the finite size corrections to be similarly good natured. Thus the results we have presented here for the probability distribution should be correct up to prefactors which might vary for the full model. This is further corroborated by a comparison of our prediction for $[q^2]_0$ with the data shown in [9]. According to equation (68), $[q^2]_0 \sim (N\epsilon^2)^{-2/3}$ for $N^{-1/2} \ll \epsilon \ll N^{-1/5}$, and this is precisely what is shown in figure 2 of [9] (the slope is not explicitly given there but graphically, $-2/3$ seems to fit very well).

5. Conclusion

We have shown that bond chaos exists in the Sherrington–Kirkpatrick model by calculating the probability distribution of the overlap q between two copies of a system, one of which has randomly perturbed bonds with respect to the other. The finite size scaling of this distribution has been calculated above, at and below the critical temperature. In the low-temperature phase four different regimes have been identified. For bond distances $\epsilon \ll N^{-1/2}$ the distribution has a variance of order 1, i.e. the equilibrium states in the two copies are still very similar. For $N^{-1/2} \ll \epsilon \ll N^{-1/5}$, the distribution is proportional to $e^{-Nc_1\epsilon^2 q^4}$, i.e. already very narrow, the width scaling as $\xi_1^{-2/3}$ with the scaling variable $\xi_1 = \sqrt{N}\epsilon$. This coincides with the scaling behaviour found in [9, 11]. For $N^{-1/5} \ll \epsilon \ll 1$, however, the scaling variable becomes $\xi_2 = N^{1/3}\epsilon$ and the distribution becomes Gaussian with width proportional to $\xi_2^{-3/2}$. For all other values of ϵ, finally, the distribution remains Gaussian, and its width goes as $N^{-1/2}$.

Acknowledgments

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